# Control of Drilling Vibrations: A Time-Delay System-Based Approach

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**Abstract:** * The main purpose of this study is the control of both axial and torsional vibrations occurring along a rotary oilwell drilling system. This work completes a previous author's paper (Boussaada et al. [2012a]) which presents the description of the qualitative dynamical response of a rotary drilling system with a drag bit, using a model that takes into consideration the axial and the torsional vibration modes of the bit. The studied model, based on the interface bit-rock, contains a couple of wave equations with boundary conditions consisting of the angular speed and the axial speed at the top additionally to the angular and axial acceleration at the bit whose contain a realistic frictional torque. Our analysis is based on the center manifold theorem and normal forms theory whose allow us to simplify the model. By this way we design two control laws allowing to suppress the undesired vibrations guaranteeing a regular drilling process.

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## 1. INTRODUCTION

Interconnected oscillatory systems often display what is called *propagation phenomena*, Hale and Lunel [1993]. In general by Lossless propagation it is understood the phenomenon associated with long transmission lines for physical signals. In engineering, this problem is strongly related to electric and electronic applications, e.g. circuit structures consisting of multipoles connected through LC transmission lines; this can also be seen in steam or gaz flows or pressures and water pipes, Niculescu [2001], Fu et al. [2006], Rasvan and Niculescu [2002]. The mathematical model is described in all these cases by a mixed initial and boundary value problem for hyperbolic partial differential equations modeling the lossless propagation. The boundary conditions are of special type, being in feedback connection with some system described by ordinary differential equations. This leads to the so-called derivative boundary conditions considered in Cooke & Krumme [1968], but also to the even more general boundary conditions of Abolina & Myshkis described by Volterra operators, see Rasvan and Niculescu [2002]. Integration along characteristics of the hyperbolic partial differential equations (here d'Alembert method) allows the association of certain system of functional equations to the mixed problem.

This paper is concerned by an application which can be modeled by such equations; therefore, the above idea is adopted, see Balanov et al. [2002], Fridman et al. [2010], Rouchon [1998], Saldivar et al. [2011]. The analysis and modeling of rotary drilling vibrations is a topic whose economical interest has been renewed by recent oilfields discoveries leading to a growing literature, see for instance Richard et al. [2007], Germay et al. [2005], Navarro-López [2009], Navarro-López and Cortés [2007], Navarro-López and Suárez [2004a], Boussaada et al. [2012a] and Navarro-López and Suárez [2004b].

Previous work by the authors Boussaada et al. [2012a] improved the modelling of vibrations of the drilling system, taking into account both axial and torsional vibrations and secondly by extending the qualitative analysis to the investigation of the nonlinear terms in the model. The use of the center manifold theorem and normal forms theory allow to the analysis of a finite dimensional approximation which conserves the main dynamics of the physical original system. Let us consider the following model for the axial vibrations $U$ and torsional vibrations $\Phi$:

$$\begin{cases} \partial_t^2 U(t,s) = c^2\,\partial_s^2 U(t,s) \\ E\,\Gamma\,\partial_s U(t,0) = \alpha \partial_t U(t,0) - H(t) \\ M\,\partial_t^2 U(t,L) = -E\,\Gamma \partial_s U(t,L) + F(\partial_t U(t,L)) \end{cases}, \quad (1)$$

and

$$\begin{cases} \partial_t^2 \Phi(t,s) = \tilde{c}^2\,\partial_s^2 \Phi(t,s) \\ G\,\Sigma\,\partial_s \Phi(t,0) = \beta \partial_t \Phi(t,0) - \Omega(t) \\ J\,\partial_t^2 \Phi(t,L) = -G\,\Sigma\,\partial_s \Phi(t,L) + \tilde{F}(\partial_t U(t,L)) \end{cases}, \quad (2)$$

where, in equation (1), $H$ is the brake motor control and $\alpha\partial_t U(t,0)$ represents a friction force of viscous type. For equation (2), the right hand side of the second equation designates the difference between the motor speed and rotational speed of the first pipe. The physical parameters of the model (1)-(2) are: $G$ is the shear modulus of the drillstring steel and $E$ the elasticity Young's modulus. Then the wave speeds can be expressed by $c = \sqrt{E/\rho}$ and $\tilde{c} = \sqrt{G/\rho}$ and $J$ the inertia $J = M\,r^2$ where $r$ is taken as the averaged radius of drillpipe and $\Gamma$ is the averaged section of the drillpipe and $\Sigma$ is the quadratic momentum. It is worth mentioning that the coupling term appears in the boundary condition of the torsional vibrations. Those parameters are taken following the numerical settings presented in the Appendix. The nonlinear aspect of the model is considered by taking functions $F$ and $\tilde{F}$ in the form: $z \mapsto pk\,z/(k^2\,z^2+\zeta)$ where the parameters $p$, $k$, $\zeta$ are some positive integer responsible of the sharpness of the top angle of the friction force graph and $p$ is some parameter deciding the amplitude of the friction force such that $0<\zeta\ll 1$ and $0<k<1$. Moreover, the behavior of the chosen friction model is close from the empirical model: the white friction force but is more handleable, which can be very useful in experimental identifications. Note also that the proposed model can be expanded to Taylor sum, which is very important when the aim is to give accurate approximation at any fixed order. The chosen functions have a close behavior to the one used in Barton et al. [2007] for modelling the friction. In a previous work by the authors, Boussaada et al. [2012a] an analytical study of the uncontrolled drilling vibrations model as functional differential equations of neutral type is established, this is done by a qualitative theory method; center manifold theorem Carr [1981] and normal forms theory Guckenheimer and Holmes [2002]. Indeed, most of the references concerned by partial differential equations (PDE) or delay differential equations (DDE) models for the drilling problem have a numerical analysis character.

This work completes the contribution of Boussaada et al. [2012a] by including some design approaches. Thus we adopt the same qualitative methodology; we reduce the considered PDE model to a singularly perturbed system of ordinary differential equations (ODE), in the goal of designing appropriate control laws allowing to suppress these vibrations, which guarantee the desired drilling process (helicoidal evolution of the bit). First we propose a PID controller, the considered spectral projection allows to reduce the PDE system to a singularly perturbed system of ODE. Secondly, motivated by the technological constraints due to the use of wireless technology which induces delay, we propose a more realistic control law which consists of delayed feedback which might be more appropriate. In this case a Bogdanov-Takens singularity (double zero eigenvalues) is considered. Furthermore, to the best of the authors knowledge, this type of singularity has never been studied for NDDE depending on parameters; thus we extend the methodology for computing the center manifold. Similar results can be found in Hale and Huang [1994] with the analysis of a physiological control model of DDE with double-zero eigenvalue singularity. This study has the same spirit as the results of Bogdanov and Takens for ODEs. We refer the reader to Kuznetsov [1998], Guckenheimer and Holmes [2002] for elements on the qualitative theory of differential equations and bifurcation theory.

The remaining part of the paper is organized as follows. The second section is concerned by preliminaries, we describe the standard procedure for reducing the PDE drillstring model to a neutral delay differential equations (NDDE). In the third section, entitled Control of drilling vibrations, we present two controllers allowing to a stable drilling process. The methodological scheme described in Ait Babram et al. [2001], Campbell [2009] is extended to the study of the parametrized model of neutral type. For the sake of self-containment, we report in the Appendix a table for the numerical settings for the parameters used in (1)-(2). We refer the reader to Boussaada et al. [2012a] for the outlines of the methodology enabling to approximate a system of NDDE by a system of ordinary differential equations (center variety) and then the study of local bifurcations (normal forms) for for further insights in FDE see Campbell [2009], Weedermann [2006], Hale and Huang [1994].

## 2. PRELIMINARIES AND PREREQUISITES

To the best of the authors knowledge, the standard procedure allowing to transform the considered PDE model to a delay system of neutral type was presented for the first time in Cooke & Krumme [1968], see also Balanov et al. [2002] and Mounier [1995]. Indeed, by using d'Alembert theorem, in Boussaada et al. [2012a] the system of PDE (1)-(2) is reduced to a system of NDDE. We adopt the normalization such that the units of length, time and torque the quantities $L$, $T = L/c$ and $E\Gamma/L$, thus system (1)-(2) is written

$$\begin{cases} \ddot{v}(t) - \dfrac{\alpha-1}{\alpha+1}\ddot{v}(t-2) = \\ -\dfrac{1}{M}\,\dot{v}(t) - \dfrac{\alpha-1}{M\,(\alpha+1)}\,\dot{v}(t-2) + \dfrac{2}{M\,(\alpha+1)}\,H(t-1) \\ +\dfrac{1}{M}F(\dot{v}(t)) - \dfrac{\alpha-1}{M\,(\alpha+1)}F(\dot{v}(t-2)) \\ \ddot{w}(t) - \dfrac{c\,E\,\Gamma\,\beta - \tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma}\ddot{w}(t-2\,\tilde{\tau}) = \\ -\dfrac{\tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,J}\,\dot{w}(t) - \dfrac{\tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,J}\dfrac{c\,E\,\Gamma\,\beta - \tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma}\,\dot{w}(t-2\tilde{\tau}) \\ +\dfrac{1}{J}\tilde{F}(\dot{v}(t)) - \dfrac{c\,E\,\Gamma\,\beta - \tilde{c}\,G\,\Sigma}{J\,(c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma)}\tilde{F}(\dot{v}(t-2\tilde{\tau})) \\ +\dfrac{2\tilde{c}\,G\,\Sigma}{J\,(c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma)}\,\Omega(t-\tilde{\tau}) \end{cases} \tag{3}$$

where $\tilde{\tau}$ is the ratio of the speeds $\tilde{\tau} = \frac{\tilde{c}}{c}$, and $v$ and $w$ are respectively the axial flat output $v(t) = U(t,L)$ and the torsional flat output $w(t) = \Phi(t,L)$.

## 3. CONTROL OF DRILLING VIBRATIONS

### *3.1 Delayed feedback controller*

Let denote by $x_1 = v$ the flat output associated with the axial vibrations, $x_2 = w$ the flat output associated with the torsional vibration, $x_3 = \dot{v}$ and by $x_4 = \dot{w}$ and consider the matrix representation of the linear part of the above system where $x=(x_1,x_2,x_3,x_4)^T$ and let set $H(t)=p_H\,v(t-1)$ and $\Omega(t)=p_\Omega\,w(t-\tau)$.

$$\begin{cases} \dot x(t) = D_1\,\dot x(t-2)+D_2\,\dot x(t-\dfrac{2\tilde c}{c})+A_0\,x(t)+A_1\,x(t-2) \\ \qquad + A_2\,x(t-\dfrac{2\tilde c}{c})+\mathcal{F}(x(t),x(t-2),x(t-\dfrac{2\tilde c}{c})) \end{cases} \tag{4}$$

where $\mathcal{F}$ is the nonlinear part of the system (3)

$$D_1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&d_{1,1,1}&0\\0&0&0&0\end{bmatrix},\ D_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&d_{2,2,2}\end{bmatrix},$$

$$A_0=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&a_{0,1,1}&0\\0&0&a_{0,2,1}&a_{0,2,2}\end{bmatrix},\ A_1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\p_H&0&a_{1,1,1}&0\\0&0&0&0\end{bmatrix},$$

$$A_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&p_\Omega&a_{2,2,1}&a_{2,2,2}\end{bmatrix}$$

where the matrices coefficients are $d_{1,1,1}=\frac{\alpha-1}{\alpha+1}$, $d_{2,2,2}=\frac{c_1E(\Gamma)\,\beta-c_2G\Sigma}{c_1E(\Gamma)\,\beta+c_2G\Sigma}$, $a_{0,1,1}=\frac{pk-\zeta}{M\zeta}$, $a_{0,2,1}=\frac{a_2k}{J\zeta}$, $a_{0,2,2}=-\frac{c_2G\Sigma}{c_1E(\Gamma)\,J}$, $a_{1,1,1}=-\frac{(\alpha-1)(\zeta+pk)}{(\alpha+1)M\zeta}$, $a_{2,2,1}=-\frac{pk(c_1E(\Gamma)\,\beta-c_2G\Sigma)}{\zeta\,(c_1E(\Gamma)\,\beta+c_2G\Sigma)J}$, $a_{2,2,2}=-\frac{c_2G\Sigma\,(c_1E(\Gamma)\,\beta-c_2G\Sigma)}{Jc_1E(\Gamma)\,(c_1E(\Gamma)\,\beta+c_2G\Sigma)}$.

Recall that in the above quoted references (concerned by PDE models), the studies were concerned only by the torsional vibrations. Thus the associated NDDE (governing the speed of such vibrations) is scalar, which is easier to study compared with (4). And for the physiological model considered in Hale and Huang [1994], $A_2=D_i=0$ for $i\in\{1,2\}$ since the model is DDE with one delay. It is worth mentioning that delay of PD controller is chosen to be the normalized proper delay of the system.

Setting the numerical values of the physical parameters given in the Appendix we have the following result

*Proposition 1.*

When $p_\Omega=p_H=0$:

- Zero is the only eigenvalue with zero real part and the remaining eigenvalues are with negative real parts. Moreover, zero is an eigenvalue of algebraic multiplicity 2 and of geometric multiplicity 1, that is zero eigenvalue is non-semisimple and the singularity is of Bogdanov-Takens, see Guckenheimer and Holmes [2002].
- The system (4) is formally stable but not asymptotically stable (although there are no characteristic roots with positive real parts).

When $p_H=-24\delta\,r$ and $p_\Omega=\frac{45\mu r^2}{10}$ for a small parameter $r$:

- The dynamics of (4) reduces on a cubic center manifold to

$$\begin{cases}\dot z_1=z_2\\ \dot z_2=\delta z_1+\mu z_2-3z_2z_1^2\end{cases} \tag{5}$$

for which the function

$$I(z)=\frac{1}{2}z_1{}^2-\frac{1}{2}\frac{z_2{}^2}{\delta}-\frac{1}{2}\delta\,z_1{}^2z_2{}^2+\frac{1}{4}\delta^2z_1{}^4+\frac{1}{4}z_2{}^4$$

is a Lyapunov function when $\delta<0$ and $\mu<0$ and then the system is globally asymptotically stable.

**Sketch of the proof:** The first assertion is obtained by establishing a linear analysis of the system. Indeed, it can be easily checked by computing the associated characteristic equation and substituting the physical values. Numerical tools as QPMR Vyhlídal & Zítek [2003] are also very useful for locating those spectral values. Let us stress on the second part of the proposition which concerns the nonlinear analysis. We follow the approach described in Hale and Huang [1994] that considers a singular delay system linearly dependent on a parameter, and in the same spirit to the decomposition established in Faria and Magalhães [1995] in the goal of computing the normal form for delay systems depending on a parameter, we extend the scheme of computing the center manifold to the case of NDDE depending on parameters and thus look for the system (4) as a perturbation of

$$\frac{d}{dt}\mathcal{D}\,x_t=\mathcal{L}_0\,x_t,\quad \text{where}\qquad \mathcal{L}_0=\mathcal{L}|_{\{p_H=0,\,p_\Omega=0\}} \tag{6}$$

Indeed, system (4) can be written as

$$\begin{aligned}\frac{d}{dt}\mathcal{D}\,x_t:&=\mathcal{L}_0\,x_t+\tilde{\mathcal{F}}(x_t)\\ &=\mathcal{L}_0\,x_t+(\mathcal{L}-\mathcal{L}_0)\,x_t+\mathcal{F}(x_t)\end{aligned} \tag{7}$$

such that

$$\mathcal{F}_{\mu,p}=\begin{bmatrix}-0.0405\,x_1^3(t)+0.0377\,x_3^3(t-2)+p_H\,x_1(t-2)\\ -1.875\,px_1^3(t)+1.874998\,px_1^3(t-1.264911064)\end{bmatrix}$$

Here we follow the theoretical schemes briefly presented in Campbell [2009], Hale and Huang [1994] and give computations steps for the equation of the evolution of the problem's solutions on the center variety for system (6).

First, we compute the basis of the generalized eigenspace corresponding to the double eigenvalue $\lambda_0=0$.

$$\Phi(\theta)^T=\begin{bmatrix}1+\theta&1+2\,\theta&1&2\\1&2&0&0\end{bmatrix},$$

where $\theta\in[-2,0]$. Recall that the adjoint linear equation associated to (4) is

$$\begin{cases}\dot u(t)=D_1\dot u(t+2)+D_2\dot u(t+\dfrac{2\tilde c}{c})\\ \qquad -A_0u(t)-A_1u(t+2)-A_2u(t+\dfrac{2\tilde c}{c})\end{cases} \tag{8}$$

with a basis for the generalized eigenspace associated to the double eigenvalue zero is given by

$$\Psi(\theta)=\begin{bmatrix}-1&-3&-7&-13\\ \xi+1&3\,\xi+2&7\,\xi+3&13\,\xi+4\end{bmatrix}.$$

Let us consider the bilinear form, see Hale and Lunel [1993]

$$
\begin{aligned}
(\psi, \varphi) =& \psi(0)(\varphi(0) - D_1\varphi(-2) - D_2\varphi(-1.264911)) \\
&+ \int_{-2}^{0} \psi(\xi + 2)A_1\varphi(\xi)d\xi \\
&+ \int_{-1.264911}^{0} \psi(\xi + 1.264911)A_2\varphi(\xi)d\xi \\
&- \int_{-2}^{0} \psi'(\xi + 2)D_1\varphi(\xi)d\xi \\
&- \int_{-1.264911}^{0} \psi'(\xi + 1.264911)D_2\varphi(\xi)d\xi.
\end{aligned} \tag{9}
$$

By using (9) we can easily normalize $\Psi$ such that $(\Psi, \Phi) = I_d$, thus the space $C$ can be decomposed as $C = P \bigoplus Q$, where $P = \{\varphi = \Phi z;\ z \in \mathbb{R}^2\}$ and $Q = \{\varphi \in C;\ (\Psi, \varphi) = 0\}$. Recall that each of those subspaces is invariant under the semigroup $T(t)$ and that the matrix $B$ (introduced in the previous section concerned by the theoretical settings) satisfying $\mathcal{A}\Phi = \Phi B$ is given by

$$
B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{10}
$$

Let us first set the following decomposition $x_t = \Phi y(t) + z(t)$ where $z(t) \in Q$ and $y(t) \in \mathbb{R}^2$, $z(t) = h(y(t))$ and $h$ is some analytic function $h : P \to Q$. Thus the explicit solution on the center manifold can be obtained by the use of the proven formula in Campbell [2009], Hale and Huang [1994] that is

$$
\dot{y}(t) = By(t) + \Psi(0)\mathcal{F}[\Phi(\theta)y(t) + h(\theta, y(t))] \tag{11}
$$

$$
\begin{aligned}
&\frac{\partial h}{dy}\{By + \Psi(0)\mathcal{F}[\Phi(\theta)y + h]\} + \Phi(\theta)\Psi(0)\mathcal{F}[\Phi(\theta)y + h] \\
&= \begin{cases} \dfrac{\partial h}{d\theta}, & -2 \le \theta \le 0 \\ \mathcal{L}(h(\theta, y)) + \mathcal{F}[\Phi(\theta)y + h(\theta, y)], & \theta = 0 \end{cases}
\end{aligned} \tag{12}
$$

where $h = h(\theta, y)$ and $\tilde{\mathcal{F}}$ is defined in (7).

Since our aim is to establish the parameter values of $p_H$ and $p_\Omega$ guaranteeing an asymptotic suppression of the vibrations after some fixed time $t_0$ we study the parameter bifurcations, the computation of the evolution equation of of the problem's solutions on the center variety for system (7) is required. In the next step, we introduce a small parameter $r$ as a scaling parameter for making a zoom into the neighborhood of the singularity. We introduce the following changes of coordinates $\left\{p_H = -24\delta\, r,\, p_\Omega = \frac{45\mu r^2}{10},\, y_1 = r z_1, y_2 = r^2 z_2\right\}$ and we scale the time by $t_{old} = r\, t_{new}$ which allows us to the following cubic normal form reduction of (4),

$$
\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = \delta z_1 + \mu z_2 - z_1^3 \end{cases}
$$

for which a normal form is given by (5). Under the negativity condition of $\delta$ and $\mu$ positive function beyond $(0, 0)$ and if $\delta < 0$ and $\mu > 0$ then $\dot{I}(z_1, z_2) = -\frac{{z_2}^2\mu}{\delta} + 3\,\frac{{z_2}^2{z_1}^2}{\delta} - \delta\, {z_1}^2{z_2}^2\mu + 3\,\delta\, {z_1}^4{z_2}^2 + {z_2}^4\mu - 3\,{z_2}^4{z_1}^2$ which is always negative. Thus the system is globally asymptotically stable and the undesired vibrations are suppressed. The proposed scheme offset the computation of a Lyapunov function for a system of PDE with nonlinear boundary conditions.

### *3.2 PID controller*

Let denote by $x_3 = v$ the flat output associated with the axial vibrations, $x_4 = w$ the flat output associated with the torsional vibration, $x_5 = \dot{v}$, by $x_6 = \dot{w}$ and $x_1 = \int_0^t x_3(s)\, ds$ and $x_2 = \int_0^t x_4(s)\, ds$ and let us consider the matrix representation of the linear part of the above system where $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ and let set $H(t) = H_p\, v(t-1) + H_d\, \dot{v}(t-1)$ and $\Omega(t) = \Omega_p w(t - \tau) + \Omega_i \int_0^{t-\tau} w(s)ds$, then the system

$$
\begin{cases} \dot{x}(t) = D_1\, \dot{x}(t-2) + D_2\, \dot{x}(t - \dfrac{2\tilde{c}}{c}) + A_0\, x(t) + A_1\, x(t-2) \\ \qquad + A_2\, x(t - \dfrac{2\tilde{c}}{c}) + \mathcal{F}(x(t), x(t-2), x(t - \dfrac{2\tilde{c}}{c})) \end{cases} \tag{13}
$$

where $\mathcal{F}$ is the nonlinear part of the system (3) with matrices

$$
D_1 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&d_{1,1,1}&0 \\ 0&0&0&0&0&0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&d_{2,2,2} \end{bmatrix},
$$

$$
A_1 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&a_{1,1,1}&0 \\ 0&0&0&0&0&0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&a_{2,2,1}&a_{2,2,2} \end{bmatrix}
$$

$$
A_0 = \begin{bmatrix} 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&H_p&0&H_d + a_{0,1,1}&0 \\ 0&\Omega_i&0&\Omega_p&a_{0,2,1}&a_{0,2,2} \end{bmatrix}
$$

where the matrices coefficients are $d_{1,1,1} = \frac{\alpha-1}{\alpha+1}$, $d_{2,2,2} = \frac{c_1 E(\Gamma)\,\beta - c_2 G\Sigma}{c_1 E(\Gamma)\,\beta + c_2 G\Sigma}$, $a_{0,1,1} = \frac{pk-\zeta}{M\zeta}$, $a_{0,2,1} = \frac{a_2 k}{J\zeta}$, $a_{0,2,2} = -\frac{c_2 G\Sigma}{c_1 E(\Gamma)\,J}$, $a_{1,1,1} = -\frac{(\alpha-1)(\zeta+pk)}{(\alpha+1)M\zeta}$, $a_{2,2,1} = -\frac{pk(c_1 E(\Gamma)\,\beta - c_2 G\Sigma)}{\zeta\,(c_1 E(\Gamma)\,\beta + c_2 G\Sigma) J}$, $a_{2,2,2} = -\frac{c_2 G\Sigma\,(c_1 E(\Gamma)\,\beta - c_2 G\Sigma)}{J c_1 E(\Gamma)\,(c_1 E(\Gamma)\,\beta + c_2 G\Sigma)}$. Setting the numerical values of the physical parameters given in the Appendix we have the following result

*Proposition 2.*
When $\Omega_p = \Omega_i = H_p = H_d = 0$:

- Zero is the only eigenvalue with zero real part and the remaining eigenvalues are with negative real parts. Moreover, zero is an eigenvalue of algebraic multiplicity 4 and of geometric multiplicity 1, that is a generalized Bogdanov-Takens singularity, see Guckenheimer and Holmes [2002].
- The system (13) is formally stable but not asymptotically stable (although there are no characteristic roots with positive real parts).

When $H_d = 13.27\, r^{11}\delta_3$, $H_p = 26.55\, r^{12}\delta_2$,

$\Omega_i = -11.47\, r^{13}\delta_1$, $\Omega_p = 11.47\, r^{10}\delta_4$ for a small parameter $r$:

- The dynamics of (13) reduces on a degree 15 center manifold to
$$\begin{cases} \dot z_1 = z_2,\ \dot z_2 = z_3,\ \dot z_3 = z_4 \\ \dot z_4 = \delta_1 z_1 + \delta_2 z_2 + \delta_3 z_3 + \delta_4 z_4 - z_4^3 \end{cases} \tag{14}$$
for which the the linear part is written as a Companion matrix,
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{bmatrix}$$
thus there exist values for $\delta_1$, $\delta_2$, $\delta_3$, and $\delta_4$ such that $A$ is Hurwitz, which guaranteeing local asymptotic stability.

**Sketch of the proof:** By the same way as for the previous control law, we use the spectral projection methodology. The singularity here is zero with algebraic multiplicity 4 and geometric multiplicity 1. A basis for the generalized eigenspace $\mathcal{M}_0$ here is
$$\Phi(\theta) = \begin{bmatrix} \frac{1}{6}\theta^3 + \theta^2 + \theta - 1 & \frac{1}{2}\theta^2 + 2\theta + 1 & \theta + 2 & 1 \\ -\frac{1}{6}\theta^3 + \frac{1}{2}\theta^2 - \theta + 2 & -\frac{1}{2}\theta^2 + \theta - 1 & -\theta + 1 & -1 \\ \theta^2 + \theta & 2\theta + 1 & 2 & 0 \\ \theta + 1 & 1 & 0 & 0 \\ 4\theta & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
The matrix $B$ associated to this type of singularity and staisfying $\mathcal{A}\Phi = B\Phi$ is given by
$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
with $\Psi$ a basis of the adjoint space satisfying $(\Psi, \Phi) = I_d$, where evaluated at zero gives
$$\Psi(0) = \begin{bmatrix} -0.00082 & -0.00082 & 0.00124 & 0.12511 & 0.01883 & 0.08717 \\ 0.01066 & 0.01066 & -0.016 & -0.14123 & -0.00384 & -0.17785 \\ 0.33261 & 0.33261 & 0.00107 & -0.69201 & -0.7673 & -0.28423 \\ 0.25042 & -0.74957 & 0.12435 & 3.20018 & 0.52271 & 2.56715 \end{bmatrix}$$
Using the following changes of coordinates ($r$ is a sufficiently small parameter)
$$\{H_d = 13.27\, r^{11}\delta_3,\ H_p = 26.55\, r^{12}\delta_2,\ \Omega_i = -11.47\, r^{13}\delta_1,$$
$$\Omega_p = 11.47\, r^{10}\delta_4,\ y_1 = r^5 z_1, y_2 = r^4 z_2,\ y_3 = r^3 z_3, y_4 = r^2 z_4\}$$
and an appropriate scaling of time allows us to the reduced system (14).

For the complete proofs see for instance Boussaada et al. [2012b] the full version of the present paper.

## 4. CONCLUDING REMARKS

The main purpose of this paper is the design of control laws allowing to suppress the undesired vibrations using a FDE qualitative analysis based approach. We establish the appropriate gains of a PID controller as well as the gains of a delayed feedback controller. Moreover, in the case of delayed feedback controller we arrive to establish a Lyapunov function guaranteeing global stability for the central dynamics, which guarantees local asymptotic stability for the initial PDE and permits to achieve the desired dynamic for the drilling process. From a realistic point of view we suggest the delayed feedback controllers since the data collection comes delayed due to the use of wireless technology which induces a delay $\hat\tau = 2.2\, s$.


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## APPENDIX

### *4.1 Graphical Illustration*

The projection of the dynamics on the center manifold for the critical value $\delta = \mu = 0$ is given in Figure 1 and $\delta = \mu = \pm 1$ is given respectively in Figure 2 and 3. Figure 4 gives the state profile $z_1$ (against time t) for various values of $\delta$ and $\mu$.

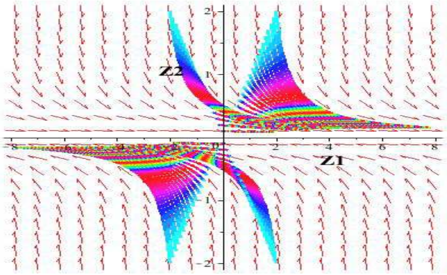


Fig. 1. Phase portrait of the system $(5)|_{\delta=\mu=0}$

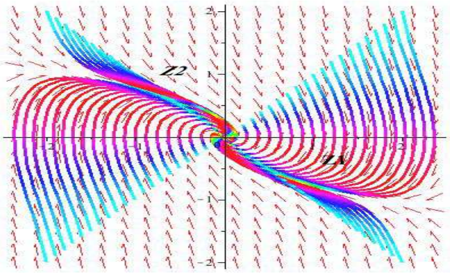


Fig. 2. Phase portrait of the system $(5)|_{\delta=\mu=-1}$

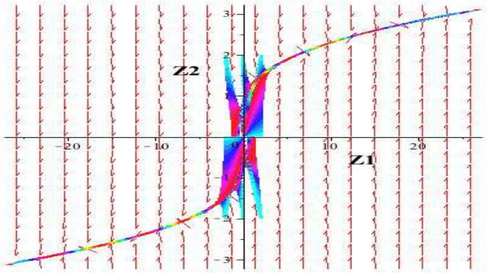


Fig. 3. Phase portrait of the system $(5)|_{\delta=\mu=1}$

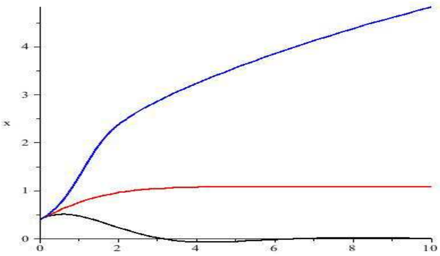


Fig. 4. The State response $z_1$ for (5) (red) $\delta = \mu = 0$, (black) $\delta = \mu = -1$ and (blue) $\delta = \mu = 1$

### *4.2 Numerical Settings*

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| G | 80 $GPa$ | E | 200 $GPa$ |
| $\rho$ | 8000 $Kg/m^3$ | r | 6 cm |
| $\Gamma$ | $35\,cm^2$ | $\Sigma$ | $19\,cm^4$ |
| L | $3000\,m$ | M | $40000\,Kg$ |
| $\alpha$ | 28 $kg/s$ | $\beta$ | 0.02 $Nms$ |
| $k$ | 0,3 | $\zeta$ | 0,01 |
| $p$ | 6 | | |